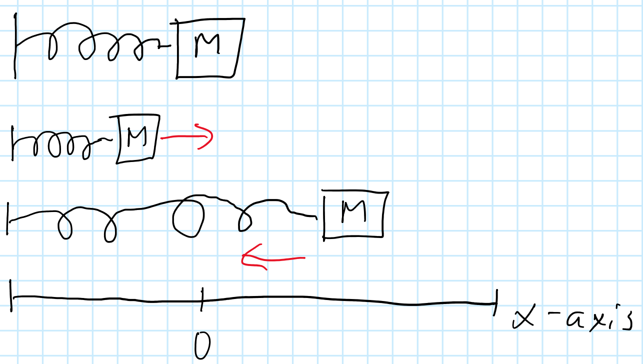
Outline:

- Simple Harmonic motion
- Linear independence of functions
- Linear differential eqn of order 2
- Forced harmonic oscillator
- Inhomogeneous equation of order 2

Simple Harmonic motion

Consider a spring on a frictionless surface attached to a wall and a mass.



Assign the 0 point along the x-axis to be the stationary natural resting point ~~of~~ the spring.

Assume that the force on the mass is proportional to how stretched or compressed the spring is.

$$F = -kx$$

From physics, $F = ma$, where F = force
m = mass
a = acceleration

$$\left.\begin{array}{l} x = \text{location} \\ \frac{dx}{dt} = \text{velocity} \\ \frac{d^2x}{dt^2} = \text{acceleration} \end{array}\right\} \quad -kx = m\frac{d^2x}{dt^2}$$

$\frac{d^2x}{dt^2}$ = acceleration

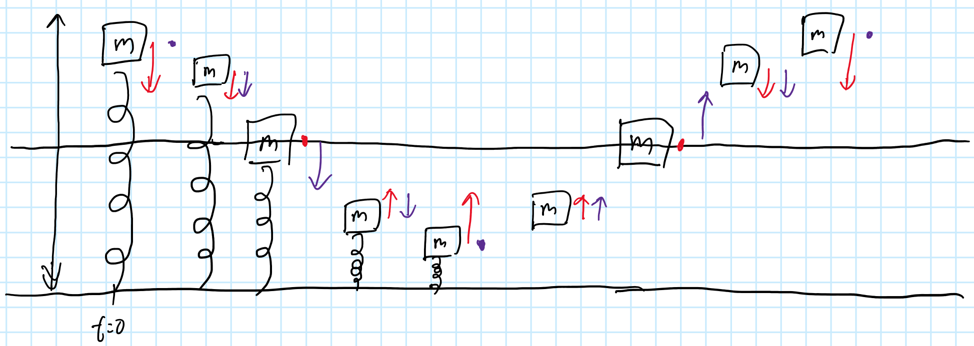
Let $\frac{k}{m} = w_0^2$, where w_0 is a constant.

Then, we get $\frac{d^2x}{dt^2} + w_0^2 x = 0$,

the ODE describing free undamped motion.

What are solutions to this ODE?

Let's plot x w.r.t. time, and draw both force and momentum



Guess: $x(t) = 0$, $x(t) = \sin(\omega_0 x)$, $x(t) = \cos(\omega_0 x)$

$\frac{dx}{dt} = 0$ $\qquad \frac{dx}{dt} = \omega_0 \cos(\omega_0 x)$ $\qquad \frac{dx}{dt} = -\omega_0 \sin(\omega_0 x)$

$\frac{dx}{dt} = 0$ $\qquad \frac{d^2x}{dt^2} = -\omega_0^2 \sin(\omega_0 x)$ $\qquad \frac{d^2x}{dt^2} = -\omega_0^2 \cos(\omega_0 x)$

Note: We can add together and multiply solutions.

$$x(t) = c_1 \sin(\omega_0 x) + c_2 \cos(\omega_0 x)$$
$$x'(t) = c_1 \omega_0 \cos(\omega_0 x) - c_2 \omega_0 \sin(\omega_0 x)$$
$$x''(t) = -c_1 \omega_0^2 \sin(\omega_0 x) - c_2 \omega_0^2 \cos(\omega_0 x) = -\omega_0^2 x.$$

We are going to learn how to solve 2nd/higher order ODEs.

Linear independence of functions

Def. A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ defined on a common interval I, is called linearly independent on I, if $\exists$ $c_1, \dots, c_n$ constants not all zero, s.t.

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0, \qquad \text{(i.e. a linear combination)}$$

for every x in I. Otherwise, they are linearly dependent.

Mentimeter
- $x, -2x, 3x, 4x$
- x^p, x^q, $p \neq q$, $x \neq 0$
- e^{px}, e^{qx}, $p \neq q$
- $e^x, 0, \sin x, 1$
- $\cos x, \sin x$

Suppose x^p and x^q are linearly dependent.
Then $\exists\, c_1, c_2$ s.t. $c_1 x^p + c_2 x^q = 0$
WLOG, $c_1 \neq 0$. So $\underbrace{x^{p-q}}_{\text{not constant}} = \underbrace{-\frac{c_2}{c_1}}_{\text{constant}}$.

$\times$ contradiction

$\Rightarrow$ x^p and x^q are linearly independent.

Linear differential equation of order n.

Thm. (Tenenbaum 19.2)

$$f_n(t)x^{(n)} + f_{n-1}(t)x^{(n-1)} + \cdots + f_1(t)x' + f_0(t)x = Q(t),$$

where $f_i(t)$ and $Q(t)$ are continuous functions on an interval I, and $f_n(t) \neq 0$ when $x \in I$, has a unique solution

$x = x(t)$ satisfying initial conditions $x(t_0) = x_0$, $x'(t_0) = x_1, \dots, x^{(n-1)}(t_0) = x_{n-1}$,

where $x_0 \in I$ and $x_0, \dots, x_{n-1}$ are constants.

If you convert this to a system of first order equations, you can apply Picard-Lindelöf to prove this

Thm (Tenenbaum 19.3)

If $f_0(t), f_1(t), \dots, f_n(t)$ and $Q(t)$ are continuous functions of t on I and $f_n(t) \neq 0$ for $t \in I$, then

(1.) The homogeneous linear ODE of order n

$$f_n(t)x^{(t)} + \dots + f_1(t)x' + f_0(t)x = 0$$

has n linearly independent solutions $x_1(t), \dots, x_n(t)$.

Ex. $\ddot{x}(t) + \omega_0^2 x(t) = 0$, and solutions $\sin(\omega_0 t)$, $\cos(\omega_0 t)$

Note, adding 0 makes it linearly dependent.

(2.) The linear combination

$$x_c(t) = c_1 x_1(t) + \dots + c_n x_n(t)$$

forms an n-parameter set of solutions of the homogeneous ODE.

(in fact, this is a general solution to the ODE, containing all possible soln)

Ex. $x_c(t) = c_1 \sin(\omega_0 t) + c_2 (\omega_0 t)$

(3) If $x_p(t)$ is a particular solution to the inhomogeneous ODE

$$f_n(t)x^{(t)} + \dots + f_1(t)x' + f_0(t)x = Q(x),$$

then $x(t) = x_c(t) + x_p(t)$ is an n-parameter family of solutions to the inhomogeneous ODE.

(where $x_c(t)$ was an n-parameter family of solutions to the homogeneous ODE, and is known as the complementary function of the inhomogeneous ODE)

Ex. $\ddot{x}(t) + \omega_0^2 x(t) = \sin(t)$ (i.e. Forced undamped spring)

Guess $x_p = A \sin(t) + B\cos(t)$ (method of undetermined coefficients; to be explained later)

$\dot{x}_p = A\cos(t) - B\sin(t)$

$\ddot{x}_p = -A\sin(t) - B\cos(t)$

$$A\sin t + B\cos t - \omega_0^2 A\sin t - \omega_0^2 B\cos t = \sin t$$

$$(\sin(t))(A - \omega_0^2 A) + (\cos t)(B - \omega_0^2 B) = \sin t$$

$$A - \omega_0^2 A = 1 \Rightarrow A = \frac{1}{1-\omega_0^2}$$

$$B - \omega_0^2 B = 0 \Rightarrow B = 0 \quad \text{if } \omega_0^2 \neq 1.$$

$$\Rightarrow x_p = \frac{1}{1-\omega_0^2}\sin(t)$$

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Then $x = x_c + x_p = c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t) + \frac{1}{1-\omega_0^2}\sin(t)$

is a 2-parameter family of solutions.

Why can we take linear combinations of solutions to the homogeneous linear ODE?

Ex from algebra $\quad x(t) + y(t) + t z(t) = 0$

Solutions: $\quad x = t,\ y = -t,\ z = 0$

$+)\quad x = 0,\ y = t\sin t,\ z = -\sin t$

Sum: $\quad x = t,\ y = -t + t\sin t,\ z = -\sin t \quad \leftarrow$ still a solution

Because it is a linear, you can break up the solution into its components, each of which adds up to 0.

Similarly, you can multiply a solution, since that's just multiplying both sides of the equation.

i.e. $kx(t) + ky(t) + ktz(t) = 0$, so if $x(t)$, $y(t)$, $z(t)$ are solutions, so is $kx(t)$, $ky(t)$, $kz(t)$.

This is the principle of superposition.

Claim: If $x_1(t)$ is a solution to

$$f_n(t)x^{(n)}(t) + \cdots + f_1(t)x'(t) + f_0(t)x(t) = Q_1(t)$$

and $x_2(t)$ is a solution to

$$f_n(t)x^{(n)}(t) + \cdots + f_1(t)x'(t) + f_0(t)x(t) = Q_2(t),$$

then $x_1(t) + x_2(t)$ is a solution to

$$f_n(t)x^{(n)}(t) + \cdots + f_1(t)x'(t) + f_0(t)x(t) = Q_1(t) + Q_2(t)$$

proof. $f_n(t)\left[x_1^{(n)}(t) + x_2^{(n)}(t)\right] + \cdots + f_1(t)\left[x_1'(t) + x_2'(t)\right] + f_0(t)\left[x_1(t) + x_2(t)\right]$

$= \left[f_n(t)x_1^{(n)}(t) + \cdots + f_1(t)x_1'(t) + f_0(t)x_1(t)\right] + \left[f_n(t)x_2^{(n)}(t) + \cdots + f_1(t)x_2'(t) + f_0(t)x_2(t)\right]$

$= Q_1(t) + Q_2(t)$

$$= \left[f_n(t) x_1^{(n)}(t) + \cdots + f_1(t) x_1'(t) + f_0(t) x_1(t) \right] + \left[f_n(t) x_2^{(n)}(t) + \cdots + f_1(t) x_2'(t) + f_0(t) x_2(t) \right]$$
$$= Q_1(t) + Q_2(t)$$

$\square$

Note that prop. 3 above is a corollary to the principle of superposition. It is straight-forward to show the other facts given above using similar arguments.

If we want an n-family parameter of solutions to a linear homogeneous ODE, we just need to find n independent solutions.

Homogeneous Linear ODE with constant coefficients

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0.$$

Recall first-order linear ODE $\quad a_1 x' + a_0 x = 0 \Rightarrow x' = -\frac{a_0}{a_1} x$

$$\Rightarrow x(t) = C e^{-a_0/a_1 \, t}$$

Let's guess that higher-order linear ODEs have solutions that look similar : $x(t) = e^{mt}$, for some constant m.

Plugging in this guess, we get

$$a_n m^n e^{mt} + a_{n-1} m^{n-1} e^{mt} + \cdots + a_1 m e^{mt} + a_0 e^{mt} = 0.$$

Dividing by e^{mt}, we get the characteristic equation of the ODE

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0,$$

and the LHS is known as the characteristic polynomial.

When can we solve algebraicly for m?

Mentimeter : Always

Only if $n \leq 5$

Only if $n \leq 4$ — Abel-Ruffini theorem

Only if $n \leq 2$

Only if $n \leq 1$

But for now, suppose we can find the roots of the characteristic equation

But for now, suppose we can find the roots of the characteristic equation

Let $m_1, \dots, m_n$ be the roots (with multiplicity) of the characteristic polynomial.

Then $e^{m_1 t}, e^{m_2 t}, \dots, e^{m_n t}$ are solutions to the ODE.

If the roots are distinct, then these are n linearly ind. solutions, giving us an n-parameter family of solutions

Ex.

$$y''' + 2y'' - y' - 2y = 0 \quad , \quad y = y(x)$$

Characteristic equation: $m^3 + 2m^2 - m - 2 = 0$

$$(m+2)(m^2 - 1) = 0$$

$$(m+2)(m+1)(m-1) = 0$$

So roots are $m = -2, -1, 1$.

Thus, e^{-2x}, e^{-x}, e^{x} are a linearly independent set of solutions.

So $y_c = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x}$ is a 3-parameter family of solutions.

Ex.

$$y'' - 3y' + 2y = 0 \quad , \quad y = y(x), \quad y(0) = 1, \quad y'(0) = 0$$

Char eqn. $m^2 - 3m + 2 = 0$

$m = 1, 2 \quad \Rightarrow \quad e^{x}, e^{2x}$ are solutions and

$$y_c = c_1 e^{x} + c_2 e^{2x}$$

Solve for IVP:

$$y_c' = c_1 e^{x} + 2c_2 e^{2x} \quad ,$$

$$y_c'' = c_1 e^{x} + 4c_2 e^{2x} \quad ,$$

$$y(0) = 1 = c_1 + c_2$$

$$y'(0) = 0 = c_1 + 2c_2$$

$$c_1 + c_2 = 1$$

$$c_1 + 2c_2 = 0$$

$$\Rightarrow c_1 = 2 \quad , \quad c_2 = -1$$

$$\Rightarrow \quad y(x) = 2e^{x} - e^{2x}.$$

Suppose that the roots are complex, and unique.
The above still works, but our solutions will be complex, and sometimes we want real roots.

Ex. Simple harmonic motion

$$\ddot{x}(t) + \omega_0^2 x(t) = 0$$

Char eqn: $m^2 + \omega_0^2 = 0$

$$m = \pm i\omega_0$$

Two linearly ind. complex solutions $e^{i\omega_0 t}$, $e^{-i\omega_0 t}$

Can we find an alternate basis?

Recall Euler's formula $e^{it} = \cos t + i \sin t$

$$\Rightarrow e^{i\omega_0 t} = \cos(\omega_0 t) + i \sin(\omega_0 t)$$

$$e^{-i\omega_0 t} = \cos(-\omega_0 t) + i \sin(-\omega_0 t)$$

$$= \cos(\omega_0 t) - i \sin(\omega_0 t)$$

Note: $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ span the same space, but are real!

So we can choose them instead, and get a 2-parameter family

$$y_c = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

In general, complex roots of polynomials with real coefficients always come in conjugate pairs $\alpha \pm \beta i$.

Lemma: The pair of linearly independent complex functions $\{e^{(\alpha+\beta i)x}, e^{(\alpha-\beta i)x}\}$ span the same space as the pair $\{e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x\}$.

proof.

$$e^{(\alpha+\beta i)x} = e^{\alpha x} e^{\beta i x} = e^{\alpha x}(\cos\beta x + i \sin\beta x)$$

$$e^{(\alpha-\beta i)x} = e^{\alpha x} e^{-\beta i x} = e^{\alpha x}(\cos\beta x - i \sin\beta x).$$

The claim is evidently true given that we can write the old pair as linear combinations of the new pair $\square$

Thus, if we can find unique real or complex roots to the characteristic polynomial of a homogeneous linear ODE with constant coefficients, we can solve the ODE.

Next time: repeated roots and inhomogeneous linear ODE with constant coefficients